Analytical formulas for gravitational lensing: higher order calculation

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We extend to higher order a recently published method for calculating the deflection angle of light in a general static and spherically symmetric metric. Since the method is convergent we obtain very accurate analytical expressions that we compare with numerical results.

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I. INTRODUCTION

In a recent paper, [1] we have introduced a new method for the calculation of the deflection angle of light in a general static and spherically symmetric metric in general relativity. This approach allows one to convert the integral for the angle into a geometrically convergent series, whose terms can be analytically calculated. Because of the strong rate of convergence of the series, typically few terms are already sufficient to obtain highly accurate analytical results. For this reason, previous analysis [1] was focussed on the calculation of first order, and a general formula, valid for an arbitrary static and spherically symmetric metric, was derived. The series is not a typical perturbative expansion in some small physical parameter, and therefore is valid also in proximity of the "photon sphere".

A good description of the deflection angle close to the photon sphere is particularly important for the study of strong gravitational lensing, a topic which has received wide attention in the recent past: for example strong gravitational lensing in a Schwarzschid black hole has been considered by Frittelli, Kling and Newman [2] and by Virbhadra and Ellis [3]; Virbhadra and collaborators have also treated the strong gravitational lensing by naked singularities [4] and in the presence of a scalar field [5]; Eiroa, Romero and Torres[6] have described Reissner-Nordström black hole lensing, while Bhadra has considered the gravitational lensing due to the GMGHS charged black hole [7]; Bozza has studied the gravitational lensing by a spinning black hole [8]; Whisker [9] and Eiroa [10] have considered strong gravitational lensing by a braneworld black hole; still Eiroa [11] has recently considered the gravitational lensing by an Einstein-Born-Infeld black hole; Sarkar and Bhadra have studied the strong gravitational lensing in the Brans-Dicke theory[12]; finally Perlick [13] has obtained an exact gravitational lens equation in a spherically symmetric and static spacetime and used to study lensing by a Barriola-Vilenkin monopole and by an Ellis wormhole.

To describe the strong gravitational lensing regime Bozza has introduced and analytical method based on a careful description of the logarithmic divergence of the deflection angle which allows to discriminate among different types of black holes, but which can only be used in proximity of the photon sphere[14]. On the other hand, other methods, which are very precise in the weak lensing regime have also been developed: for example, Mutka and Mähönen [15, 16] and Belorobodov [17] have derived improved formulas for the deflection angle in a Schwarzschild metric, while Keeton and Petters[18] have introduced a formalism for computing corrections to lensing observables in a static and spherically symmetric metric beyond the weak deflection limit. These methods fail to describe the strong lensing regime and do not even reproduce a photon sphere. This is in contrast with our method of [1], which already to first order predicts a photon sphere and provides much better results than usual perturbative expansions.

The purpose of this paper is to further improve the description of the strong lensing regime extending previous analysis [1] to higher orders and providing with a general formula for the systematic and analytical calculation of the deflection angle in any static spherically symmetric metric.

The paper is organized as follows: in section II we review our method. In section III we obtain the general expression for the higher order contributions. In section IV we discuss the application of the series to a few examples. Finally in section V we draw our conclusions.

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II. THE METHOD

In this section we outline the method introduced earlier [1].We consider a general static and spherically symmetric metric which corresponds to the line element

$$ds^2 = B(r)dt^2 - A(r)dr^2 - D(r)r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \ . \tag{1}$$

Notice that the Schwarzschild metric is a special case of this equation. Although at this stage we leave the metric coefficients $A(r)$, $B(r)$ and $D(r)$ unspecified, we assume that $\lim_{r\to\infty} f(r) = 1$, where $f(r) = (A(r), B(r), D(r))$ which are consistent with a flat spacetime at infinity.

One can express the angle of deflection of light propagating in this metric in terms of the integral [19]

$$\Delta\phi = 2\int_{r_0}^{\infty} \sqrt{A(r)/D(r)} \sqrt{\left[\left(\frac{r}{r_0}\right)^2 \frac{D(r)}{D(r_0)} \frac{B(r_0)}{B(r)} - 1\right]^{-1}} \frac{dr}{r} - \pi \ , \tag{2}$$

where r_0 is the distance of closest approach of the light to the center of the gravitational attraction.

In general it is possible to evaluate analytically Eq. (2) only in a limited number of cases, depending upon the form of the metric. Nonetheless we wish to prove that precise analytical formulas can still be obtained even though no explicit expression for the exact integral may exist.

As explained in [1] one can introduce the function

$$V(z) \ \equiv \ z^2 \frac{D(r_0/z)}{A(r_0/z)} - \frac{D^2(r_0/z)\ B(r_0)}{A(r_0/z)B(r_0/z)D(r_0)} + \frac{B(r_0)}{D(r_0)} \ , \tag{3}$$

where $z = r_0/r$ and thus convert Eq. (2) into the form

$$\Delta\phi = 2\int_0^1 \frac{dz}{\sqrt{V(1) - V(z)}} - \pi \ . \tag{4}$$

This expression bears a close resemblance to the expression for the period of a classical oscillator with a potential $V(z)$. It was shown in [1] that an efficient way of dealing with such integral is to use the nonperturbative method based on the Linear Delta Expansion (LDE), developed by Amore and collaborators [20, 21].

Let us review the main aspects of the method: we interpolate the full potential $V(z)$ with a simpler potential $V_0(z)$, which should be chosen in such a way that the integral inside Eq. (4) can be performed explicitly when $V(z)$ is substituted by $V_0(z)$. Then, we write

$$V_\delta(z) \equiv V_0(z) + \delta(V(z) - V_0(z)) \ ,$$

where δ is a dummy expansion parameter that is chosen equal to unity at the end of the calculation. The reference potential $V_0(z)$ may depend upon a set of adjustable parameters, which we will collectively call λ. It is sufficient for present purposes to consider the simplest reference potential $V_0(z) = \lambda z^2$ and rewrite the deflection angle as

$$\Delta\phi_\delta = 2\int_0^1 \frac{dz}{\sqrt{E_0 - V_0(z)}} \frac{1}{\sqrt{1 + \delta\Delta(z)}} - \pi \ , \tag{5}$$

where $E_0 = V_0(1) = \lambda$,

$$\Delta(z) \equiv \frac{E - V(z)}{E_0 - V_0(z)} - 1 \ . \tag{6}$$

and $E = V(1)$. Clearly Eq. (5) reduces to the exact expression Eq. (4) when $\delta = 1$.

If $|\Delta(z)| < 1$ for $0 \le z \le 1$ the expansion of equation (5) in powers of δ yields a perturbation series that converges towards the exact result. It was shown in earlier papers [1, 20, 21] that this condition is met when λ is greater than a critical value, $\lambda > \lambda_C$. Thus, the sequence of partial sums converges towards the exact result that is independent λ, although each partial sum depends on the adjustable parameter. In order to minimize the dependence of the approximate results on λ we resort to the the Principle of Minimal Sensitivity (PMS)[22] which in present case reads:

$$\frac{\partial}{\partial\lambda} \Delta\phi^{(N)} = 0 \ , \tag{7}$$

where $\Delta\phi^{(N)}$ is the partial sum of order N.

This condition determines an optimal value of λ around which the truncated series is less sensitive to changes in that parameter. In addition to it, one expects that the resulting series exhibits improved convergence properties [1]. More precisely, it has been verified that the series obtained by means of the PMS exhibit geometric rate of convergence providing markedly accurate results with just a few terms [1]. In fact, just a first order calculation proved sufficient for the prediction of the location of the photon sphere with remarkable accuracy [1].

Previous calculations [1] also show that the rate of convergence of the PMS series depends critically upon the physical parameters of the model, and in particular the accuracy of the results deteriorates as r_0 approaches the photon sphere. For this reason in this paper we are interested in the systematic improvement of our results by means of general and simple formulas that facilitate the calculation of any desired order of approximation for a given arbitrary static and spherically symmetric metric.

III. HIGHER ORDER CALCULATION

Under the assumption that the general metric considered here is flat at the infinity we can express the potential in Eq. (3) as

$$V(z) = \sum_{n=1}^{\infty} v_n z^n \ . \tag{8}$$

Substitution of this expression into Eq. (6) yields

$$\Delta(z) = \sum_{n=1}^{\infty} \frac{v_n}{\lambda} \sum_{k=0}^{n-1} \frac{z^k}{1+z} - 1 \ . \tag{9}$$

Additionally, the angle of deflection becomes

$$\Delta\phi = 2\int_0^1 \frac{dz}{\sqrt{E_0 - V_0(z)}} \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} \delta^n (-1)^n \Delta^n(z) - \pi \equiv \sum_{n=0}^{\infty} \Delta_n \ , \tag{10}$$

where the meaning of the terms Δ_n is obvious.

Assuming that $|\Delta(z)| < 1$ for $z \in (0,1)$ and the series is convergent, then summation and integration can be interchanged, thus leading to

$$\Delta\phi = 2\sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{(2n-1)!!(-1)^j}{2^n j!(n-j)!\lambda^{j+1/2}} \ \delta^n \sum_{r_1 \ r_2 \ \dots \ r_j}^{\infty} v_{r_1} v_{r_2} \dots v_{r_j} \ \Omega^{(j)}_{r_1 r_2 \dots r_j} - \pi \tag{11}$$

where the rank–N tensors

$$\Omega^{(N)}_{n_1 n_2 \dots n_N} \equiv \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\cdots\sum_{k_N=0}^{n_N-1} \int_0^1 \frac{z^{k_1+k_2+\dots+k_N}}{(1+z)^N} \frac{dz}{\sqrt{1-z^2}} \ . \tag{12}$$

are dimensionless and completely symmetric in the lower indices. The explicit form of these tensors up to order four is given in the Appendix.

By induction one derives the general expression

$$\begin{aligned} \Omega^{(k)}_{n_1 \dots n_k} \ &= \ \frac{2^{k-1}}{(2k-1)!!}\sqrt{\pi} \left\{ \frac{\Gamma(\sum_{i=1}^k n_i/2 + 1/2)}{\Gamma(\sum_{i=1}^k n_i/2 - (k-1))} - \sum_{P_j} \frac{\Gamma(\sum_{i\neq j}^k n_i/2 + 1/2)}{\Gamma(\sum_{i\neq j}^k n_i/2 - (k-1))} \right. \\ &+ \left. \sum_{P_{j_1 j_2}} \frac{\Gamma(\sum_{i\neq j_1 j_2}^k n_i/2 + 1/2)}{\Gamma(\sum_{i\neq j_1 j_2}^k n_i/2 - (k-1))} + \dots \right\} \ . \end{aligned} \tag{13}$$

where $\sum_{P_{j_1 j_2 \dots j_l}}$ means sum of all the different combinations of the $k-l$ elements chosen out of k total elements. The amount of terms is simply given by the binomial coefficient $\begin{pmatrix} k \\ k-l \end{pmatrix} = \frac{k!}{l!(k-l)!}$.

A crucial observation is that the indices of Ω take all possible values when the sum is performed. Consequently we cannot distinguish among the possible different combinations inside (13) and the same result must be obtained taking a single one with the proper multiplicity factor given by the binomial. We can therefore make the substitution $\Omega^{(k)}_{n_1 \dots n_k} \to \sum_{l=1}^{k} \tilde{\Omega}^{(k|l)}_{n_1 \dots n_l}$ where

$$\tilde{\Omega}^{(k|l)}_{n1 \dots n_l} \equiv \frac{2^{k-1}}{(2k-1)!!} \sqrt{\pi} \begin{pmatrix} k \\ k-l \end{pmatrix} (-1)^{k-l} \frac{\Gamma(\sum_i^l n_i/2 + 1/2)}{\Gamma(\sum_i^l n_i/2 - (k-1))} \ . \tag{14}$$

By substitution of this expression into Eq. (11) we obtain the final expression

$$\Delta\phi = 2 \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(2n-1)!!(-1)^j}{2^n j!(n-j)! \lambda^{j+1/2}} \ \delta^n \sum_{r_1 \ r_2 \ \dots \ r_j}^{\infty} v_{r_1} v_{r_2} \dots v_{r_j} \sum_{l=1}^{j} \tilde{\Omega}^{(j|l)}_{r_1 \dots r_l} - \pi \ , \tag{15}$$

In the end, we can explicitly write the expression corresponding to the first few orders as

$$\Delta\phi = -\pi + \frac{315}{64\sqrt{\lambda_{PMS}}} \Sigma_0 - \frac{105}{16 \lambda_{PMS}^{3/2}} \Sigma_1 + \frac{189}{32 \lambda_{PMS}^{5/2}} \Sigma_2 - \frac{45}{16 \lambda_{PMS}^{7/2}} \Sigma_3 + \frac{35}{64 \lambda_{PMS}^{9/2}} \Sigma_4 + \dots \tag{16}$$

where

$$\Sigma_j \equiv \sum_{r_1 \ r_2 \ \dots \ r_j}^{\infty} v_{r_1} v_{r_2} \dots v_{r_j} \sum_{l=1}^{j} \tilde{\Omega}^{(j|l)}_{r_1 \dots r_l} \ . \tag{17}$$

For simplicity, in Eq. (16) we have selected the optimal parameter of *first order* $\lambda = \lambda_{PMS}$ for all orders of approximation. The reason is that it is not possible to obtain analytical expressions of λ_{PMS} for greater orders, and that it is sufficient that $\lambda_{PMS} > \lambda_C$ for the series to be geometrically convergent. Thus, in all the applications below we choose the value of λ_{PMS} obtained previously [1]:

$$\lambda^{(1)}_{PMS} = \frac{\Sigma_1}{\Sigma_0} = \frac{2}{\sqrt{\pi}} \sum_n v_n \frac{\Gamma(n/2 + 1/2)}{\Gamma(n/2)} \ . \tag{18}$$

IV. APPLICATIONS

In this section we consider some applications of the general formula (15). The first example is the Schwarzschild metric given by

$$B(r) = A^{-1}(r) = \left(1 - \frac{2GM}{r}\right) \quad , \quad D(r) = 1 \ , \tag{19}$$

where M is the Schwarzschild mass and G the gravitational constant. The angle of deflection of a ray of light reaching a minimal distance r_0 from the black hole is given Eq. (4), and the exact result can be expressed in terms of incomplete elliptic integrals of the first kind[23] as

$$\Delta\phi = 4 \sqrt{\frac{\overline{r}_0}{\Upsilon}} \ \left[F\left(\frac{\pi}{2}, \kappa\right) - F\left(\varphi, \kappa\right) \right] \ , \tag{20}$$

where $\overline{r}_0 \equiv r0/GM$ and

$$\Upsilon \equiv \sqrt{\frac{\overline{r}_0 - 2}{\overline{r}_0 + 6}} \quad , \quad \kappa \equiv \sqrt{(\Upsilon - \overline{r}_0 + 6)/2\Upsilon} \quad , \quad \varphi \equiv \sqrt{\arcsin\left[\frac{2 + \Upsilon - \overline{r}_0}{6 + \Upsilon - \overline{r}_0}\right]} \ . \tag{21}$$

The "potential" for this metric is particularly simple since it is just a polynomial of third degree:

$$V(z) = z^2 - \frac{2GM}{r_0} z^3 \ . \tag{22}$$

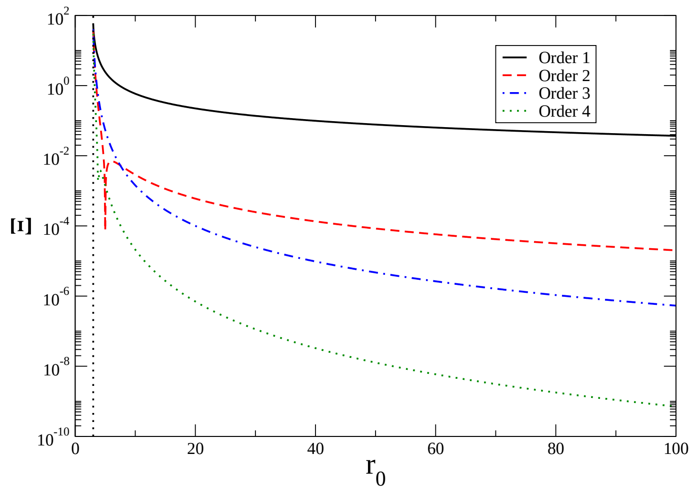

FIG. 1: Error of the deflection angle as a function of r_0 for the Schwartzchild metric with $GM = 1$. The dotted vertical line shows the location of the photon sphere. (color online)

In order to apply Eq. (15) we choose $GM = 1$ and calculate the error of the deflection angle as a function of the distance of maximal approach r_0 $(= \overline{r}_0)$, defined as

$$\Xi = \left| \frac{\Delta\phi_{PMS} - \Delta\phi_{exact}}{\Delta\phi_{exact}} \right| \times 100 \ , \tag{23}$$

where $\Delta\phi_{exact}$ is the Darwin solution (20). Fig. 1 shows that the errors of the PMS approximations through fourth order are extremely small and smaller that one percent even close to the photon sphere located at $r_0 = 3$ (the one percent error is obtained at $r_0 \approx 3.2$ for the fourth–order partial sum).

Present approach yields similar results for the Reissner-Nordström (RN) metric

$$B(r) = A^{-1}(r) = \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) \quad , \quad D(r) = 1 \ . \tag{24}$$

that describes a black hole with charge. In this case Eiroa, Romero and Torres [6] have been able to express the deflection angle in terms of elliptic integrals of the first kind (see eqn. (A3) of [6]).

The "potential" for this example is also polynomial:

$$V(z) = z^2 - \frac{2GM}{r_0} z^3 + \frac{Q^2}{r_0^2} z^4 \ . \tag{25}$$

The behavior of the error displayed in Fig. 2 is similar to that observed in the case of the Schwarzschild metric.

Both examples discussed above lead to polynomial potentials in z and are amenable to exact solutions in terms of elliptic functions. However, it should be noticed that our method applies to more general cases that cannot be solved exactly.

In an earlier paper [1] we considered the propagation of light in a charged black hole coupled to Born-Infeld electrodynamics [11], which corresponds to the effective metric

$$A(r) = \frac{\sqrt{\omega(r)}}{\psi(r)} \quad , \quad B(r) = \sqrt{\omega(r)}\psi(r) \quad , \quad D(r) = \frac{1}{\sqrt{\omega(r)}} \ , \tag{26}$$

where

$$\omega(r) = 1 + \frac{Q^2 b^2}{r^4} \tag{27}$$

$$\psi(r) = 1 - 2\frac{M}{r} + \frac{2}{3b^2} \left\{ r^2 - \sqrt{r^4 + b^2 Q^2} + \frac{\sqrt{|bQ|^3}}{r} \ F\left[\arccos\left(\frac{r^2 - |bQ|}{r^2 + |bQ|}\right), \frac{1}{\sqrt{2}}\right] \right\} \ . \tag{28}$$

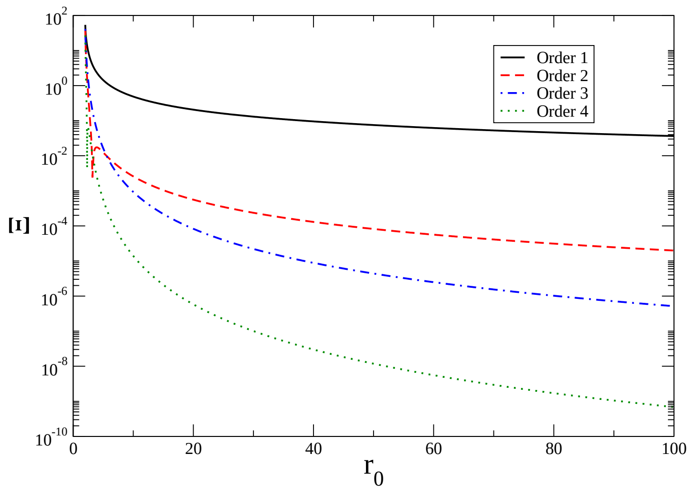

FIG. 2: Same as in Fig. 1 for the RN metric with $GM = 1$ and $Q = 1$. (color online)

TABLE I: Deflection angle for the Born-Infeld metric with $GM = 1$, $Q = 1/2$ and $b = 1$.

r_0	Numerical	PMS_1	PMS_2	PMS_3	PMS_4
3	5.04066558	3.9836332	4.769690116	4.695977905	4.961794995
4	1.989492408	1.905459664	1.988326336	1.985415327	1.989578734
5	1.302767649	1.274857482	1.302841883	1.302239811	1.302781935
10	0.4908767995	0.4881229478	0.4908905773	0.4908704512	0.4908768924
50	0.08300041129	0.08293664219	0.08300048071	0.08300040744	0.0830004113
100	0.04073430348	0.04071929339	0.04073431165	0.04073430326	0.04073430348

Here, $F(a, b)$ is the incomplete elliptic integral of first kind. We follow the convention of Ref. [11] and set $G = 1$.

In this case one obtains the potential

$$V(z) \equiv z^2 \frac{\psi(r_0/z)}{\omega(r_0/z)} - \frac{\psi(r_0)\omega(r_0)}{\omega^2(r_0/z)} + \psi(r_0)\omega(r_0) \tag{29}$$

which can be expanded around $z = 0$ as

$$V(z) = \sum_{n=2}^{\infty} v_n z^n \ . \tag{30}$$

Clearly, in this case the accuracy of our results will depend not only upon the order of the partial sum but also on the number of Taylor coefficients used to approximate the potential. We illustrate this point in Fig. 3 for the deflection angle when $GM = 1$, $Q = 1/2$ and $b = 1$, and for $r_0 = 3$, which is very close to the photon sphere where the angle diverges. This situation clearly corresponds to a strong gravitational lensing regime. The horizontal axis shows the number of Taylor coefficients of the potential used in the calculation; the reader will appreciate that just a small number of them ($n \approx 4$) is sufficient for reasonable accuracy. For larger values of n the results do not appear to improve noticeably and a sort of plateau is reached. We point out that the main advantage of our approach is that it leads to relatively simple analytical functions of the parameters of the metric.

Table I shows the deflection angles in terms of r_0 for the Born-Infeld metric with the same parameters as before. The potential $V(z)$ is approximated with a Taylor polynomial of order 10 and the calculations are performed with our method through order 4.

As a final example we consider the metric of Weyl gravity [24, 25, 26, 27]:

$$B(r) = A^{-1}(r) = \left(1 - \frac{2\beta}{r} + \gamma r - kr^2\right) \quad , \quad D(r) = 1 \ , \tag{31}$$

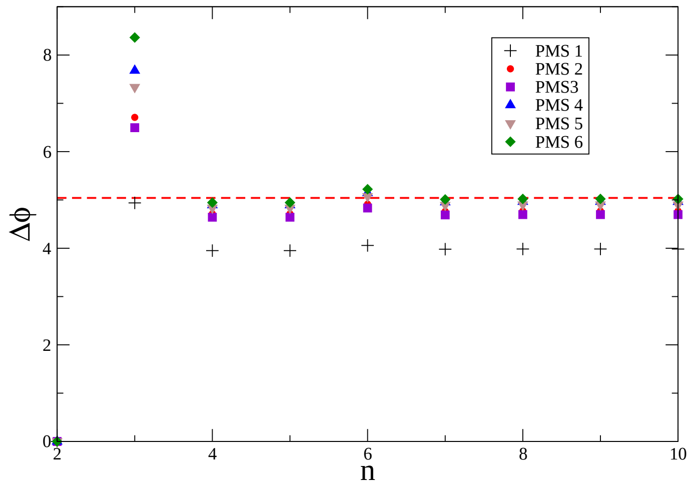

FIG. 3: Deflection angle for the Born Infeld metric with $GM = 1$, $Q = 1/2$, $b = 1$, and $r_0 = 3$. The horizontal axis specifies the order of the Taylor approximation to the potential. The dashed line is the numerical result. (color online)

where β,γ and k are constants. When $\beta = GM$ and γ and k small enough one recovers the Schwarzschild metric on a certain distance scale. The linear and quadratic terms are significant only on galactic and cosmological scales, if the corresponding parameters are sufficiently small.

The potential corresponding to this metric is once again polynomial in z:

$$V(z) = -kr_0^2 + \gamma r_0 z + z^2 - \frac{2\beta}{r_0} z^3 \ . \tag{32}$$

Notice that the constant term cancels out of the integral for the deflection angle that will therefore be independent of k.

On defining $\Lambda \equiv \sqrt{1 - \frac{8\beta}{\pi r_0} + \frac{2\gamma r_0}{\pi}}$ we obtain the fourth–order expression

$$\Delta\phi^{(4)} = -\pi + \sum_{n=0}^{4} \frac{d_{2n+1}}{\Lambda^{2n+1}} \ , \tag{33}$$

where

$$d_1 = 3.19285 \tag{34}$$

$$d_3 = \frac{0.42847\beta}{r_0} - 0.10350 \tag{35}$$

$$d_5 = \frac{1.02342\beta^2}{r_0^2} - \frac{0.48610\beta}{r_0} + 0.058569 \tag{36}$$

$$d_7 = \frac{0.75158\beta^3}{r_0^3} - \frac{0.55570\beta^2}{r_0^2} + \frac{0.13858\beta}{r_0} - 0.01166 \tag{37}$$

$$d_9 = \frac{1.15417\beta^4}{r_0^4} - \frac{1.18963\beta^3}{r_0^3} + \frac{0.46370\beta^2}{r_0^2} - \frac{0.08094\beta}{r_0} + 0.00533 \ . \tag{38}$$

Expanding this equation in powers of β and γ we obtain

$$\Delta\phi^{(4)} \approx \frac{4\beta}{r_0} - \gamma r_0 + \left(\frac{15\pi\beta^2}{4r_0^2} - \frac{4\beta^2}{r_0^2} + \gamma\beta - \frac{3\ \gamma\pi\beta}{2} + \frac{\gamma^2 r_0^2}{2} \right) + \dots \ , \tag{39}$$

where the term between parenthesis contains the contribution of second order in the expansion. Notice that the first two terms are those usually reported in the literature [25, 27].

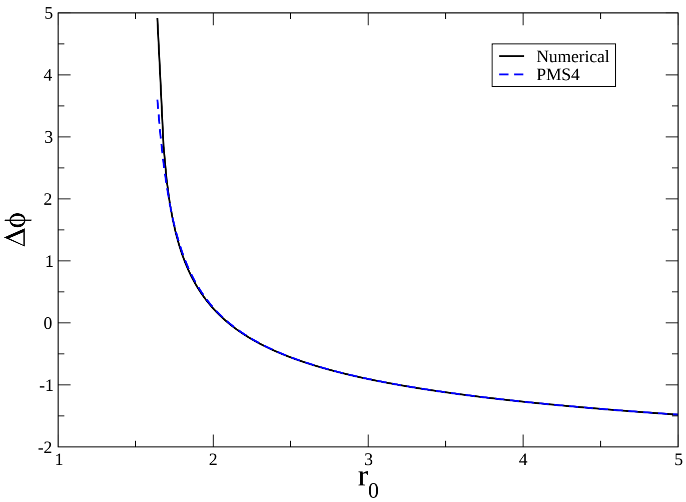

FIG. 4: Deflection angle for the Weyl metric with $\beta = \gamma = 1$ for r_0 close to the photon sphere. (color online)

Fig. 4 shows results for the deflection angle given by the Weyl metric with $\beta = \gamma = 1$. Our fourth–order analytical formula agrees remarkably well with the numerical calculation.

Notice that the deflection angle displays an interesting behavior depending upon the sign of γ: as we see in Fig. 4 for $\gamma > 0$ the angle is negative at large distances while it becomes positive close to the photon sphere. Using our analytical formula we are able to provide an accurate expression for the value of r_0 corresponding to absence of deflection:

$$r_\star \approx 2\sqrt{\frac{\beta}{\gamma}} + \frac{\beta\left(-\beta(40960 + 9\pi(112 + 75\pi(-16 + 3\pi)))\gamma \ + 864\sqrt{\beta}\pi(-16 + 5\pi)\sqrt{\gamma} + 1024(32 - 9\pi)\right)}{32768} \ . \qquad (40)$$

In the case of Fig. 4 this formula predicts the location of $r_\star \approx 2.08$, which is remarkably close to the exact numerical value.

In the opposite regime, $\gamma < 0$, the deflection angle is real only on a finite region that we obtain from

$$\lambda_{PMS} = 1 - \frac{8\beta}{\pi r_0} + \frac{2\gamma r_0}{\pi} \ . \qquad (41)$$

In fact, when $\beta > 0$ and $\gamma > -\frac{\pi^2}{64\beta}$ then $\lambda_{PMS} > 0$ only for

$$\frac{\sqrt{64\beta\gamma + \pi^2}}{4\gamma} - \frac{\pi}{4\gamma} < r_0 < -\frac{\sqrt{64\beta\gamma + \pi^2}}{4\gamma} - \frac{\pi}{4\gamma} \ . \qquad (42)$$

In Fig. 5 we compare our fourth–order analytical result with the numerical one for the Weyl metric with $\beta = 1$ and $\gamma = -\pi^2/128$. The agreement is remarkable for all the values of r_0 considered.

V. CONCLUSIONS

In this paper we have generalized a method developed earlier [1] for obtaining analytical expressions for the deflection angle of light travelling in an arbitrary static and spherically symmetric metric. We have applied a new higher–order formula to several examples confirming the accuracy suggested by a rigorous proof and estimate of the rate of convergence [1].

We believe that our formalism is particularly useful for several reason: first of all, because it stands on a firm mathematical ground; second, because it is completely general and can be applied with little or no effort to any such metric; finally because the analytic results obtained with our method could provide a valuable tool for the analysis of different models. Just to mention one example, in the case of the Weyl metric, we have been able to derive an analytical expression for the absence of deflection for a ray of light travelling in a metric with $\beta > 0$ and $\gamma > 0$.

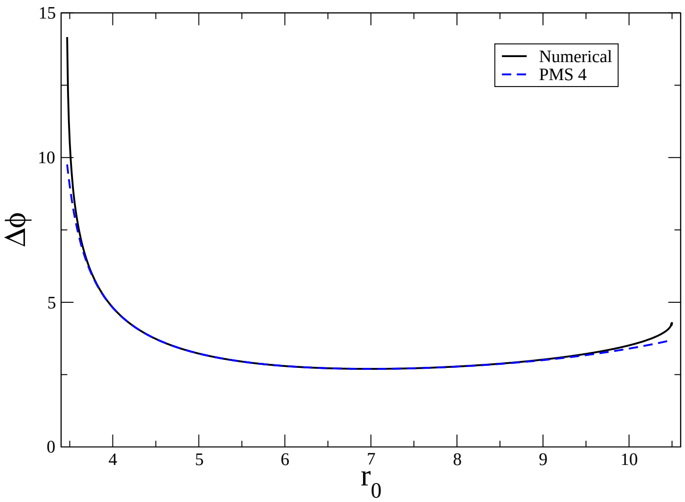

FIG. 5: Deflection angle for the Weyl metric with $\beta = 1$ and $\gamma = -\pi^2/128$.

APPENDIX A: $\Omega^{(j)}_{n_1\ldots n_j}$

Explicit expressions of the symmetric tensor Ω through order four, which one obtains by straightforward algebra:

$$\Omega^{(0)} = \frac{\pi}{2} \tag{A1}$$

$$\Omega^{(1)}_{n} = \sqrt{\pi}\frac{\Gamma(n/2+1/2)}{\Gamma(n/2)} \tag{A2}$$

$$\Omega^{(2)}_{n_1 n_2} = \frac{2}{3}\sqrt{\pi}\left(-\frac{\Gamma\left(\frac{n_2+1}{2}\right)}{\Gamma\left(\frac{n_2}{2}-1\right)} - \frac{\Gamma\left(\frac{n_1+1}{2}\right)}{\Gamma\left(\frac{n_1}{2}-1\right)} + \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2-2)\right)}\right) \tag{A3}$$

$$\begin{aligned}\Omega^{(3)}_{n_1 n_2 n_3} = \frac{4}{15}\sqrt{\pi}\Bigg(&\frac{\Gamma\left(\frac{n_1+1}{2}\right)}{\Gamma\left(\frac{n_1}{2}-2\right)} + \frac{\Gamma\left(\frac{n_2+1}{2}\right)}{\Gamma\left(\frac{n_2}{2}-2\right)} + \frac{\Gamma\left(\frac{n_3+1}{2}\right)}{\Gamma\left(\frac{n_3}{2}-2\right)} - \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2-4)\right)} \\ &- \frac{\Gamma\left(\frac{1}{2}(n_1+n_3+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_3-4)\right)} - \frac{\Gamma\left(\frac{1}{2}(n_2+n_3+1)\right)}{\Gamma\left(\frac{1}{2}(n_2+n_3-4)\right)} + \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+n_3+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2+n_3-4)\right)}\Bigg)\end{aligned} \tag{A4}$$

$$\begin{aligned}\Omega^{(4)}_{n_1 n_2 n_3 n_4} = \frac{8}{105}\sqrt{\pi}\Bigg(&-\frac{\Gamma\left(\frac{n_1+1}{2}\right)}{\Gamma\left(\frac{n_1}{2}-3\right)} - \frac{\Gamma\left(\frac{n_2+1}{2}\right)}{\Gamma\left(\frac{n_2}{2}-3\right)} - \frac{\Gamma\left(\frac{n_3+1}{2}\right)}{\Gamma\left(\frac{n_3}{2}-3\right)} - \frac{\Gamma\left(\frac{n_4+1}{2}\right)}{\Gamma\left(\frac{n_4}{2}-3\right)} \\ &- \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2-6)\right)} + \frac{\Gamma\left(\frac{1}{2}(n_1+n_3+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_3-6)\right)} + \frac{\Gamma\left(\frac{1}{2}(n_2+n_3+1)\right)}{\Gamma\left(\frac{1}{2}(n_2+n_3-6)\right)} \\ &+ \frac{\Gamma\left(\frac{1}{2}(n_1+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_4-6)\right)} + \frac{\Gamma\left(\frac{1}{2}(n_2+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_2+n_4-6)\right)} + \frac{\Gamma\left(\frac{1}{2}(n_3+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_3+n_4-6)\right)} \\ &- \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+n_3+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2+n_3-6)\right)} - \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2+n_4-6)\right)} \\ &- \frac{\Gamma\left(\frac{1}{2}(n_1+n_3+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_3+n_4-6)\right)} - \frac{\Gamma\left(\frac{1}{2}(n_2+n_3+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_2+n_3+n_4-6)\right)} + \frac{\Gamma\left(\frac{1}{2}(n_1+n_2+n_3+n_4+1)\right)}{\Gamma\left(\frac{1}{2}(n_1+n_2+n_3+n_4-6)\right)}\Bigg) \;.\end{aligned} \tag{A5}$$

APPENDIX B: Σ_k

Explicit expressions for the coefficients Σ_k defined in Eq. (17) for the Taylor expansion of an arbitrary potential through order three:

$$\Sigma_0 = \frac{\pi}{2} \tag{B1}$$

$$\Sigma_1 = v_1 + \frac{1}{2}\pi v_2 + 2v_3 + \dots \tag{B2}$$

$$\begin{aligned}\Sigma_2 &= \frac{2v_1^2}{3} + 2v_2v_1 + \pi v_3 v_1 - \frac{2}{3}v_3\ v_1 + \frac{1}{2}\pi v_2^2 \\ &+ \frac{5}{4}\pi v_3^2 - \frac{4v_3^2}{3} + 4\ v_2v_3 + \dots\end{aligned} \tag{B3}$$

$$\begin{aligned}\Sigma_3 &= \frac{7v_1^3}{15} + 2v_2v_1^2 + \frac{12}{5}v_3v_1^2 + 3v_2^2\ v_1 - \frac{3}{2}\pi v_3^2 v_1 \\ &+ \frac{47}{5}v_3^2v_1 + 3\pi v_2v_3\ v_1 - 2v_2v_3v_1 + \frac{1}{2}\pi v_2^3 - \frac{3}{2}\pi\ v_3^3 \\ &+ \frac{122v_3^3}{15} + \frac{15}{4}\pi v_2v_3^2 - 4v_2v_3^2 + 6v_2^2v_3 + \dots\end{aligned} \tag{B4}$$

$$\begin{aligned}\Sigma_4 &= \frac{12v_1^4}{35} + \frac{28}{15}v_2v_1^3 + \frac{76}{35}v_3v_1^3 + 4\ v_2^2v_1^2 + 3\pi v_3^2v_1^2 \\ &- \frac{124}{35}v_3^2v_1^2 + \frac{48}{5}\ v_2v_3v_1^2 + 4v_2^3v_1 + 9\pi v_3^3v_1 \\ &- \frac{708}{35}v_3^3\ v_1 - 6\pi v_2v_3^2v_1 + \frac{188}{5}v_2v_3^2v_1 \\ &+ 6\pi v_2^2\ v_3v_1 - 4v_2^2v_3v_1 + \frac{1}{2}\pi v_2^4 + \frac{99}{16}\pi\ v_3^4 - \frac{104v_3^4}{7} \\ &- 6\pi v_2v_3^3 + \frac{488}{15}v_2\ v_3^3 + \frac{15}{2}\pi v_2^2v_3^2 - 8v_2^2v_3^2 + 8v_2^3v_3 + \dots\end{aligned} \tag{B5}$$

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